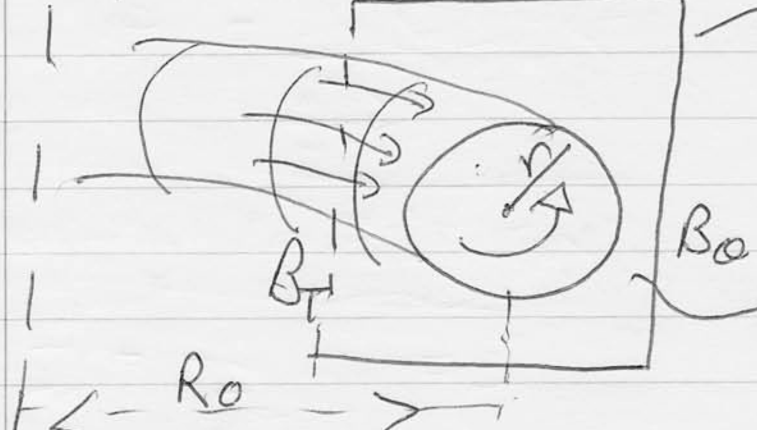
(Case Study)
Aside: Magnetic Field Lines on a Tokamak: A Practical Example of Phase Space Evolution on Tori

→ What is a Tokamak?

- toroidal confinement device for magnetized plasma



→ primary of transformer

$\left(\frac{R_0}{r} \gg 1\right)$

→ plasma as secondary of transformer

$I = I_0 \Rightarrow B_T(R) = 2I_0/cR$ → toroidal field (external)

$B_T \gg B_\theta \qquad B_\theta(r) = \int_0^r r'dr' \frac{J_T}{c}(r')$ → poloidal field (plasma current)

- $B_\theta(r)$ → shorts out charge separation due ∇B drift
  → stability, confinement
  → heating (ohmic)

more info: "Tokamak Plasma, A Complex Physical System" R.B. Kadomtsev

Minimal Model: The "Toroidal Cow" for Toroidal Field Configurations

- tokamak as periodic cylinder with:

$$\left.\begin{array}{l} 0<r<a \\ L = 2\pi R_0 \end{array}\right\} \left\{\begin{array}{l} B_z = B_T \quad \text{(uniform, external)} \\ B_\theta(r) \end{array}\right.$$

$$\langle \underline{B} \rangle = B_\theta(r)\,\hat{\theta} + B_z \hat{z}$$

$$\underline{B} = \langle \underline{B} \rangle + \underline{\tilde{B}}_\perp$$

- field line: $$\frac{dz}{B_T} = \frac{r\,d\theta}{B_\theta(r) + \tilde{B}_\theta} = \frac{dr}{\tilde{B}_r}$$

$\therefore$

$$\frac{d\theta}{dz} = \frac{1}{r}\,\frac{B_\theta(r) + \tilde{B}_\theta}{B_z} \cong \frac{1}{r}\,\frac{B_\theta(r)}{B_z} \qquad \tilde{B}_\theta \ll B_\theta(r)$$

$$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_z}$$

For un-perturbed field configuration:

$$\frac{d\theta}{dz} = \frac{1}{r}\,\frac{B_\theta(r)}{B_z} = \frac{1}{R\,q(r)} \;; \qquad \underbrace{q(r)}_{\text{safety factor}} \equiv B_z r / R B_\theta(r)$$

$$\frac{dr}{dz} = 0 \qquad \text{(no radial wandering)}$$

$\iota(r) \equiv$ winding rate (i.e. rotational transform)
(# poloidal circuits per toroidal)

$\rightarrow$ Relation to Hamiltonian Dynamics ?.

$\frac{dx}{dz} = \frac{\tilde{B}_x}{B_0}$, $\frac{dy}{dz} = \frac{B_y(x) + \tilde{B}}{B_0}$ $\quad \underline{\nabla} \cdot \underline{B} = 0 \Rightarrow$ Ham.

Useful to observe similarity between:

a) Hamiltonian System with:

$$H = H(x,y) \quad \text{so} \quad \begin{cases} \dot{x} = -\partial H / \partial y \\ \dot{y} = \partial H / \partial x \end{cases} \qquad \left(\underline{\nabla} \cdot \underline{V}_{ph} = 0 \right)$$

so Liouville Eqn. for $f(t, x, y)$ is:

$$\frac{\partial f}{\partial t} + \dot{x} \frac{\partial f}{\partial x} + \dot{y} \frac{\partial f}{\partial y} = 0$$

$$\therefore \quad \frac{\partial f}{\partial t} - \frac{\partial H}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial H}{\partial x} \frac{\partial f}{\partial y} = 0$$

Can further specialize: $H = H_0(x) + \tilde{H}(x,y)$

$$\Rightarrow \quad \begin{aligned} \dot{x} &= -\partial \tilde{H} / \partial y \\ \dot{y} &= \frac{\partial H_0}{\partial x} + \frac{\partial \tilde{H}}{\partial x} \end{aligned}$$

and

$$\frac{\partial f}{\partial t} + \frac{\partial H_0}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \tilde{H}}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial \tilde{H}}{\partial x}\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + V_y(x)\frac{\partial f}{\partial y} + \{\tilde{H}, f\} = 0$$

e.g. $\begin{cases} H = \phi(r, \theta) \\ \text{G.C. plasma} \end{cases}$

b) Equation for Magnetic Flux $\psi(r, \theta)$

$\underline{B} = B_0 \hat{z} + \underline{\nabla}\psi \times \hat{z}$ → $\underline{B}$ field

$\psi = A_z(r, \theta)$

$\psi = \langle \psi(r) \rangle + \tilde{\psi}(r, \theta)$ → Magnetic flux function

then, by definition:

$\underline{B} \cdot \underline{\nabla}\psi = 0$ (flux constant along magnetic field lines)

$\Rightarrow$

$$\left(B_0 \frac{\partial}{\partial z} + \frac{B_\theta(r)}{r}\frac{\partial}{\partial \theta} + \underline{\tilde{B}_\perp} \cdot \nabla_\perp \right)\psi = 0$$

$\Rightarrow$

$$\left(\frac{\partial}{\partial z} + \frac{1}{R q(r)}\frac{\partial}{\partial \theta} + \frac{\tilde{B}_\perp \cdot \nabla_\perp}{B_0} \right)\psi = 0$$

, can read off analogy: (isomorphism)

$$\begin{cases} z \leftrightarrow t, \quad r \leftrightarrow x, \quad r\,d\theta \leftrightarrow y \end{cases}$$

$$\begin{cases} 1/R\,q(r) \leftrightarrow V_y(x) \leftrightarrow \omega(I) \\ \dfrac{\partial \omega(I)}{\partial I} \neq 0 \Rightarrow q'(r) \neq 0 \quad \text{"shear"} \end{cases}$$

(winding rate varies with radius)

$$\begin{cases} \langle V_y(x) \rangle \leftrightarrow B_\theta(r) \end{cases}$$

$$\begin{cases} \tilde{B}_\perp \leftrightarrow \nabla \tilde{H} \times \hat{z} \end{cases}$$

$$\begin{cases} \underline{\nabla}_\perp \cdot \underline{\tilde{B}}_\perp = 0 \\ \nabla \cdot (\nabla \tilde{H} \times \hat{z}) = 0 \end{cases}$$

Liouville Thm.
($\underline{\nabla} \cdot \underline{B} = 0$ underlies Hamiltonian structure)

Thus, Hamiltonian trajectory on 2-torus in phase space (for 2 degs. freedom) equivalent to trajectory of magnetic field line on torus of radius (minor) $= r$ in space!

$$\begin{cases} r \leftrightarrow I \\ 1/R\,q(r) \leftrightarrow \omega(I) \\ \theta \leftrightarrow \Theta \quad \text{(angle variable)} \end{cases}$$

An Observation

Consider solution of flux equation perturbatively

i.e.

$$\underline{B} = \underline{B}_0 + \tilde{\underline{B}} \qquad \underline{B}_0 = B_0 \hat{z} + B_\theta \hat{\theta}$$

$$\psi = \langle \psi(r) \rangle + \tilde{\psi}$$

$\therefore$

$$\underline{B} \cdot \nabla \psi = 0 \Rightarrow$$

$$(\underline{B}_0 \cdot \underline{\nabla}) \tilde{\psi} = -\tilde{B}_r \frac{\partial}{\partial r} \langle \psi(r) \rangle$$

expand $\psi, \tilde{B}_r$ as:

$$\tilde{B}_r = \sum_{m,n} \tilde{B}_r(r)_{\substack{m\\n}} e^{i(m\theta - n\phi)}$$

$$z \to R\phi$$

$\Rightarrow$

$$\left(-\frac{in}{R} B_0 + \frac{im}{r} B_\theta \right) \tilde{\psi}_{\substack{m\\n}} = -\tilde{B}_{r\,\substack{m\\n}} \frac{\partial \langle \psi(r) \rangle}{\partial r}$$

$$\tilde{\psi}_{m,n}(r) = \frac{-\tilde{B}_{r\,\substack{m\\n}}(r)\, \partial \langle \psi(r) \rangle / \partial r}{-i \frac{B_0}{R} \left(n - \frac{m}{q(r)} \right)}$$

$$\therefore \quad \tilde{\psi}_{m,n}(r) = \frac{i R \left(\tilde{B}_{r\,\substack{m\\n}}(r)/B_0\right) \partial \langle \psi(r) \rangle / \partial r}{\left(n - \frac{m}{q(r)}\right)} \Rightarrow \text{small divisor problem!}$$

$\Rightarrow$ perturbative solution diverges at
$q(r) = m/n$ $\Rightarrow$ defines resonant surface (special tori)

i.e. radius where pitch of field line ($q(r)$) resonates with pitch of perturbation (m/n)

$\Rightarrow$ linear solution to Liouville Eqn. fails here.
$\Downarrow$
$\underline{n} \cdot \underline{\omega} = 0$
(resonances)

Welcome to small divisor problem!